Ex-ante Implications of Sovereign Default

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February 28, 2014

Abstract

I study how default decisions affect incentives for international consumption risk-sharing in a small open economy model. Risk-sharing reduces the exposure of domestic consumption to country-specific productivity shocks. However, default decisions have an option value which comes from the randomization over ex-post default regimes, and therefore depends on the exposure to shocks. I show that this inherent trade-off can lead to endogenous risk-taking: Even if the country is risk-averse and full insurance against productivity shocks is possible, the optimal plan may keep consumption volatile because of the option value of default. I relate the value of the default option to the external debt to capital ratio, and identify threshold effects that determine whether risk-sharing or risk-taking is optimal.

Key words: financial integration, consumption risk-sharing, threshold effects

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1 Introduction

I propose a simple model of a small open economy to study some potential implications of sovereign default on other capital allocation decisions. Sovereign defaults are a recurring feature of many developing and emerging economies. For example, countries like Argentina, Brazil, Myanmar and Sri Lanka have defaulted thrice between 1980 – 2002. Reinhart and Rogoff (2004) suggest that the empirical regularity of default in many developing economies suggests that default is a conscience policy instrument, which can therefore have feedback effects on other capital allocation decisions.\(^1\)

To study some implications of sovereign default decisions, I introduce the possibility of default into an otherwise standard model of international risk-sharing. A small open economy makes capital allocation decisions both before and after the realization of a productivity shock. Ex-ante, the country decides how much to borrow from international lenders. Access to international capital is limited by a simple borrowing constraint. The country can invest capital either domestically or in a perfectly negatively correlated foreign sector. I assume that the representative consumer is risk-averse, and so without default, the optimal capital allocation uses the foreign sector to fully diversify against productivity shocks. To study how default can impact diversification decisions, I introduce the possibility of default on debt after the realization of productivity shocks. I propose a simple punishment mechanism for default in the context of the model, whereby a country that defaults is forced to produce ex-post in a safe but strictly dominated (traditional) sector.

The option to default changes substantively the optimal allocation of capital both ex-ante and ex-post. Specifically, I show that there are two regimes. If access to capital markets is limited by a tight borrowing constraint, default is never realized in equilibrium (because of the punishment mechanism) and the ex-ante capital allo-

\(^{1}\)In particular, Reinhart and Rogoff (2004) argue that while default events can be detached from fundamentals by well-known coordination problems (currency crisis, bank runs, sudden stops, etc.), the “serial default” observed in some emerging economies is much more systematic and that “sovereign defaults tend to recur like clockwork in some countries, while being entirely absent in others” (Reinhart and Rogoff, 2004).
cation is the same as without default. However, if the country has sufficient access to capital markets, the country borrows more, does not diversify risks and defaults on debt when it realizes a low productivity shock. As a result, the expected productivity of capital decreases (because capital is allocated to the less productive traditional sector when default occurs), and the variance of productivity increases (because the country does not use the foreign sector to diversify against productivity shocks). Surprisingly, perhaps, a decrease in expected productivity and an increase in volatility occur in equilibrium despite the fact that the country has to pay a premium to fully compensate international lenders for the default-risk.

There is a simple underlying intuition for the trade-off between ex-ante diversification and the ex-post decision about default. International diversification provides a hedge against country-specific productivity shocks. On the other hand, the option value of default is related to the randomization over different ex-post regimes, where default is optimal when productivity is low and not optimal when productivity is high. Since diversification reduces exposure to productivity shocks, it curtails randomization and decreases the option value of default. The benefits of diversification and option value of default are therefore interdependent. The optimal allocation decision then depends on a trade-off between the insurance value of diversification and the option value of default. The larger the external debt to capital ratio, the more important the option value of default becomes, leading to the threshold effect in terms of the borrowing constraint. Figure 1, illustrates this key intuition.

The results of the model provide some basic insights about trade-offs facing emerging economies as they become more integrated into international capital markets. Capital market integration provides opportunities for diversification, which can reduce exposure to country-specific productivity shocks. However, a large literature has shown that diversification and risk-sharing has not increased in emerging economies as would have been anticipated by standard models of consumption risk-sharing (See for e.g., Kose et al., 2009; Flood et al., 2009; Yeyati and Williams, 2011). One reason maybe that capital market integration has gone hand-in-hand with a large
**Figure 1:** *Option Value: Trade-off between risk sharing and risk taking*

1.1: No Risk Sharing and Option Value

1.2: Risk Sharing and Option Value

Figure 1.1 illustrates that the option value of default is related to the randomization over different *ex-post* regimes. Default in low state and not default in high state is optimal only if the agent decides not to share risk through *ex-ante* capital allocations. Figure 1.2 illustrates that diversification reduces the exposure to productivity shocks and therefore, it curtails randomization and decreases the option value of default.

increase in sovereign debt, and concomitant increase in default incentives. The model in this paper provides a simple theoretical mechanism consistent with this idea, and therefore complements the growing empirical literature on consumption risk-sharing and default in emerging economies.

The model is also related to several strands of theoretical literature. In particular, in terms of the set-up, the current paper is related to a wide literature on limited enforcement of contracts and default risk. Key references for the theory of limited enforcement without default are Kehoe and Levine (1993), Alvarez and Jermann (2000) and Kocherlakota (1996), while for limited enforcement with default see Zame (1993) and Dubey et al. (2005).

The former studies assume complete set of contingent assets to search for allocations that are efficient subject to the limited enforceability of contracts. In their
set-up, default never happens in equilibrium and therefore, this market structure may not be useful for understanding actual emerging markets which have a history of serial default. Later set of studies assume incomplete markets and in their setting default happens in equilibrium with a positive probability.

In particular, follow up work such as Arellano (2008), Aguiar and Gopinath (2005) and Bai and Zhang (2012), present models of emerging economies in which default happens in equilibrium. However, their analysis is restricted to international financial market transactions of non-contingent bonds only. The point of distinction of the current paper from this previous literature is the emphasis in this paper, on the role of the interaction of ex-post default risk with ex-ante allocation of capital in contingent assets for diversification purposes.

The main implication of the model is that capital trade between economies (for diversification) may be pareto inferior to no-trade in the presence of the ex-post default option. In terms of this implication, the paper is closest to Newbery and Stiglitz (1984). In a standard competitive international trade framework with incomplete risk markets, they show that free trade of goods is parerto inferior to autarky. The underlying mechanism at work in their model is that free trade not only changes the distribution of risk among the producers and consumers but it also effects the capital allocation decisions of the producers.

The analysis in this paper is also related to the large literature on home-bias in consumption and equities. Lack of consumption risk-sharing can be viewed as a reflection of a consumption home-bias, while lack of international capital diversification is related to (though distinct from) an equity-home bias. Sorensen et al. (2007) provide empirical evidence that both biases are closely linked, at least in industrialized countries. Lewis (1999) discusses both home-biases, the relationship between them, and various explanations proposed in the literature. Most theoretical expla-

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2The consumption home-bias refers to the empirical observation that equality of consumption growth rates – an implication of the basic common priors, Arrow-Debreu complete markets model – is dramatically rejected by the data (e.g., Backus et al., 1992). Equity home-bias refers to the empirical observation that the proportion of foreign assets held by domestic investors is too small relative to the predictions of standard portfolio theory (e.g., Levy and Sarnat, 1970; French and Poterba, 1991).
nations depend on market imperfections and/or the presence of non-tradable goods. The general idea that a consumption home-bias can be related to domestic control over ex-post policy responses is an interesting feature of the current paper.

In the next section, I present a two-period model of a small open economy with a single representative agent, and study efficient allocations as the solution to a planner’s problem. The basic set-up of the model, presented in Section 2, is simple in order to highlight the key interaction between consumption risk-sharing and the value of default options. From there on, I proceed in two steps. I first introduce the model without default in Section 3 and analyze the capital allocation decisions in this benchmark case. I then introduce default with risk premium to the benchmark set up in Section 4 and present the main result which shows how default with risk premium affects ex-ante capital allocation decisions. Proofs of the propositions are given in the Appendix.

2 Model: Basic Set-up

Utility function and Preferences:
The felicity function describing instantaneous utility is denoted as \( u : \mathbb{R}_+ \rightarrow \mathbb{R} \cup \{-\infty\} \). I assume throughout that the felicity function, \( u \), is continuous on \( \mathbb{R}_+ \), twice continuously differentiable on \( \mathbb{R}_{++} \), and of the form \( u(x) = x^{1-\gamma}/(1 - \gamma) \) which also implies that the temporaneous marginal utility is of the form \( u'(x) = x^{-\gamma} \) for some \( 0 < \gamma < 1 \). The restriction \( 0 < \gamma < 1 \) implies that the representative agent is (strictly) risk-averse.

The discount factor, \( \beta \in (0,1] \) and for simplicity and to save on the notation, I also assume that the rate of depreciation of capital, denoted by \( \delta = 1 \).

Technologies and Productivity Shocks:
All technologies are linear. The assumption of linear technologies implies that investments abroad are advantageous only for risk-sharing purposes. \(^3\) In period 1, the

\(^3\text{Alternatively, with decreasing returns to scale, international diversification could also be used to equate marginal rates of returns in countries with different capital stocks. This would complicate} \)
country has a given capital stock $\bar{K} > 0$ which it can use to produce in a traditional sector with productivity $A > 0$. There is full capital depreciation.\footnote{Full capital depreciation saves on notation. The results of the paper extend to any depreciation rate in $[0, 1]$, but including a depreciation rate adds to notation without any additional insights.} The output of production can be used for consumption in period 1 (denoted $c_1$), or accumulated as capital for the period 2 production process (denoted $K_1$).

In period 1 the planner also chooses how much capital to invest at home and how much to invest abroad. Investment abroad is only valuable as a way to diversify risks from productivity shocks in period 2. I assume that in period 2 the country is able to produce in a modern sector. Output from the modern sector is used in trade and is more valuable than output from the traditional sector, but productivity in the modern sector is subject to a shock.\footnote{The shock could be interpreted directly as a productivity shock, but could also be viewed as a shock to the domestic consumption value of domestic output coming from shocks to foreign demand for exports or exchange rate volatility.} Specifically, there are two equally likely states of the world $s \in \{H, L\}$.\footnote{The important parts of the distributional assumption on shocks are symmetry and a compact support. The assumption that there are two equally likely states of the world is otherwise not critical to the analysis. It is possible, for example, to extend the model by allowing for a continuous and uniformly distributed shock, but at considerable cost in terms of notation and complexity of the solution. With a continuous distribution, threshold values appear as endogenous limits on an integral, and it is easier therefore to gain clear insights from the model with a discrete distribution. The symmetry assumption has substantive content as well. Shocks that are not symmetric around their mean imply that either the foreign or domestic economy is inherently more productive and would therefore induce an incentive for cross-border capital flows that is not related to risk-sharing.} In state $H$ the country has productivity $\bar{a}$ in the modern sector, and in state $L$ the country has productivity $a$ in the modern sector. I assume that productivity is higher in state $H$ than state $L$, and that the modern sector dominates the traditional sector so that $\bar{a} > a > A$. Essentially, this assumption captures in reduced-form, a stylized feature of emerging economies. As trade barriers are reduced, emerging economies have been able to specialize production for foreign markets, leading to productivity gains but also greater volatility. Production gains are generally attributed to increased specialization, while greater volatility has been attributed to the effects of foreign preference and technology shocks on the domestic economy.

\begin{itemize}
\item The results of the paper extend to any depreciation rate in $[0, 1]$, but including a depreciation rate adds to notation without any additional insights.
\item The shock could be interpreted directly as a productivity shock, but could also be viewed as a shock to the domestic consumption value of domestic output coming from shocks to foreign demand for exports or exchange rate volatility.
\item The important parts of the distributional assumption on shocks are symmetry and a compact support. The assumption that there are two equally likely states of the world is otherwise not critical to the analysis. It is possible, for example, to extend the model by allowing for a continuous and uniformly distributed shock, but at considerable cost in terms of notation and complexity of the solution. With a continuous distribution, threshold values appear as endogenous limits on an integral, and it is easier therefore to gain clear insights from the model with a discrete distribution. The symmetry assumption has substantive content as well. Shocks that are not symmetric around their mean imply that either the foreign or domestic economy is inherently more productive and would therefore induce an incentive for cross-border capital flows that is not related to risk-sharing.
\end{itemize}
consumption value of output from exports.\footnote{7}

\textit{International diversification:}

The potential for risk-sharing is modeled by allowing the country to invest a fraction of its accumulated capital abroad, prior to the realization of productivity shock in the modern sector. Capital outflows are invested in a foreign, modern sector with perfectly negatively correlated productivity shocks. This means that the country can, in principle, fully diversify against the productivity shock $a_s \in \{\bar{a}, \tilde{a}\}$ in the modern sector through international diversification of capital. Denote the fraction of capital invested at home by $\theta_1 \in [0, 1]$, and the fraction invested in the foreign sector by $(1 - \theta_1)$.\footnote{8}

Two implications of the assumptions on technologies at home and abroad are worth re-emphasizing:

\begin{enumerate}
\item Since production shocks in the modern sector at home and abroad are perfectly negatively correlated, the country can perfectly diversify production risks by choosing $\theta_1 = 1/2$. Hence, when the representative agent is risk-averse, there are maximal insurance gains from risk-sharing.

\item Since technologies have constant returns to scale in all sectors (home and abroad) and both states are equally likely, there are no benefits from investment in the foreign sector other than for risk-sharing purposes. A risk neutral agent would be indifferent between investments in either sector. Hence, the focus of the analysis is purely on foreign investments for the purpose of risk-sharing.
\end{enumerate}

\footnote{7}{The advantage of specialization in sectors where a country has comparative advantages goes back to the classical Ricardo and Hecksher-Ohlin-Mundell trade models (see, e.g., Dornbusch et al., 1977). The trade-off between the productivity gains from specialization and the increased vulnerability to external shocks is established theoretically in Easterly et al. (2000), and verified empirically for emerging economies in Kose et al. (2005).}

\footnote{8}{In reality, investments in the foreign sector might generally be accompanied by a counterflow of investments into the domestic, modern sector from abroad. It is notationally simpler and without loss of generality to suppress these counterflows since they do not have welfare implications for the domestic consumer. It is important only to highlight that foreign investment into the domestic economy are not subject to default or repossession, at least not to the advantage of the domestic consumers.}
**Ex-post Capital Allocation:**

Unlike the *ex-ante* allocation decision between domestic and foreign capital investments ($\theta$), the allocation of capital between the domestic modern and domestic traditional sectors is made *ex-post* to the realization of shocks. Hence, only domestically held capital is allocated between the modern and traditional sectors in period 2. Denote the domestic capital allocated to the modern sector by $K_2^M = (K_{2H}^M, K_{2L}^M)$ and the domestic capital allocated to the traditional sector by $K_2^T = (K_{2H}^T, K_{2L}^T)$.\(^9\)

**Borrowing:**

As well as the direct (or equity) investments abroad captured by $\theta$, the open economy is connected to international capital markets through access to international borrowing and lending. In particular, I assume that the domestic economy is small and can borrow or lend capital in international capital markets at a given rate of interest $r > 0$ in period 1. The borrowing (or lending) decision is denoted by $D_1$. Borrowing is subject to an exogenous collateral constraint that ensures that countries are able to repay debts in period 2.

\[ D_1 \leq \bar{\omega}K_1 , \quad (1) \]

where $\bar{\omega}$ is the collateral constraint which is restricted to lie in the range $[0, a/(1+r)]$. Note that the restriction on $\bar{\omega}$ implies that default – when it occurs – is always voluntary since net output (after paying debt obligations) in period 2 is always at least equal to

\[ aK_1 - (1 + r)\bar{\omega}K_1 = K_1(a - (1 + r)\bar{\omega}) \geq 0 \iff \bar{\omega} \leq a/(1 + r) . \quad (2) \]

**Assumptions on productivity parameters:**

In general, no restrictions are imposed on lending and borrowing beyond the borrowing constraint in equation (1). However, to capture the interaction between the

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\(^9K_{2H}^M\) and $K_{2L}^M$ denote capital allocated domestically in the modern sector in state $H$ and state $L$, respectively. $K_{2H}^T$ and $K_{2L}^T$ denote capital allocated domestically in the traditional sector in state $H$ and state $L$, respectively.
ex-ante risk-sharing decision ($\theta$) and the ex-post default decision ($\lambda$), an additional assumption is required on the productivity parameters in the model to ensure that economy will be a borrower country.

**Assumption 1** Denote the standard deviation of the random variable $x$ by $\text{std}(x)$. The following relationships exist between the productivity parameters in the modern sector, traditional sector and the interest rate.

(A1.1) $\bar{K} > 0$, $\bar{\omega} > 0$.

(A1.2) (a) $A < \bar{a} < \bar{\bar{a}}$; (b) $A < \bar{a} < (1 + r) < \bar{\bar{a}}/2$.

(A1.3) $\text{std}(x) = (\bar{a} - \bar{a})/2 < A$.

The second assumption, (A1.2), consists of two parts. The first part of (A1.2), part (a), simply states that the modern sector dominates the traditional sector. With a weak inequality this assumption is without loss of generality since the capital allocation decision between modern and foreign sectors is made ex-post, and if $A > \bar{a}$ the economy could simply utilize the traditional sector in state $L$ (in which case $A$ essentially replaces $\bar{a}$). The assumption of a strict inequality is only required to ensure uniqueness of results and is otherwise not essential. As such, the first inequality is an innocuous assumption.

The second part of (A1.2), part (b), states that gross returns on capital in the high state are twice as high as the international cost of capital and the gross return in the low state is less than international cost of borrowing. In general, the expected productivity of capital in the emerging economy is higher than the international cost of capital, which is in turn higher than the productivity in the low state of the world. The fact that there has traditionally been a considerably greater flow of capital into emerging economies as a group than out of emerging economies, suggests that the first part is empirically relevant but quantitatively too strong. It is important to note that the quantitative requirement comes from the assumption that the probability of a low state is 0.5 (and even then it is sufficient but not necessary to ensure that
the country will borrow when it must pay a risk-premium for default). Since, in the analysis, the probability of a low state corresponds to the probability of default, the high requirement on the rate of return in the open economy relative to the world at large is dictated by the high probability of default. This high default probability is maintained for simplicity and greater clarity of the analysis, because it helps to remove investment incentives from the *ex-ante* capital allocation decision and thereby helps to focus the analysis on risk-sharing. The assumption is not essential to the intuition behind the trade-off between risk-sharing and default options in general.  

One important implication of the above assumptions coupled with linear technology and strictly increasing utility function is that the borrowing constraint is necessary for the existence of solution (to the underlying problem). In the absence of the borrowing constraint, a planner would borrow an infinite amount of $D_1$. This is because the expected return from investing the borrowed capital in the modern technology will exceed the cost of borrowing and hence would make the planner borrow an infinite amount of debt. Since the utility function is strictly increasing in $D_1$ in period 1, the utility function that follows the Inada condition will then be undefined. Finally, the last inequality, (A1.3), puts an upper bound on the variance of the technology shock relative to the productivity in the traditional sector. The recent economic history of emerging economies has numerous examples of external debt defaults (or substantial restructuring or renegotiation of debts) which, almost without exception, were precipitated by large negative output shocks. In the context of the model, $A$ should be interpreted as the productivity in the open economy following this type of default event. The inequality $\text{std}(a) = (\bar{a} - a)/2 < A$ puts a bound on how large the variation in productivity in the modern sector can be, 

\footnote{With the conditions in part (A1.2), the model in this section simply illustrates that if the return on capital is sufficiently high for emerging economies to borrow despite credit-risk premia, the trade-off between risk-sharing and default remains. Qualitatively, this does not require any change in the structure of the model, and the quantitative requirement (what constitutes a “sufficiently” high rate of return in the high state) is dictated here by other assumptions that simplify the analysis and interpretation but are not germane to the central argument. Since emerging markets have witnessed large inflows of debt over the past 20-30 years despite paying substantial credit-risk premiums, the key motivation for the assumption lies in its implications for borrowing behavior and these clearly find considerable empirical support.}
relative to the output in a default event. For example, in the limiting case where $a = A$, the bound states that the drop in output per unit of capital from best state to worst state, should not be greater than 30%. Note that this still represents an enormous output shock but it is empirically possible.\footnote{For example, during the massive downturn in Argentina around the end of 2001, output fell by 15% in one year and by just over 20% from its previous peak amidst sovereign default and devaluation. Even in Russia during the early 1990’s, the cumulative fall in GDP is estimated to have been less than 40%.

In the main Proposition of this Section, the assumption will be needed to ensure that the threshold values identified for risk-sharing and risk-taking in terms of the debt to capital ratio lie in the range $[0, a/(1 + r)]$.}

3 Efficient Allocations: Benchmark

**Timeline:**
Figure 2 gives a timeline summarizing the decision problem of the planner in the benchmark case. Recall that in period 1 the planner makes a borrowing/lending decision ($D_1$), a capital accumulation decision ($K_1$), and an *ex-ante* capital allocation decision between the home and foreign sectors ($\theta_1$). In the benchmark case, in period 2 the planner then makes a decision regarding an *ex-post* capital allocation decision between the modern and traditional sectors ($K^M_2$ and $K^T_2$). The exogenous parameters of the model are the productivity parameters $A, a, \bar{a}$, the interest rate $r$, the initial capital stock $\bar{K}$ and the borrowing constraint $\bar{\omega}$.

**Figure 2:** *Timeline Risk Sharing with Collateral Constraint and Default*

<table>
<thead>
<tr>
<th>Time</th>
<th>Period 1</th>
<th>Period 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Given</td>
<td>$K, \bar{\omega}$</td>
<td>$K_1, \theta_1, D_1$</td>
</tr>
<tr>
<td>Decisions</td>
<td>$c_1, K_1, \theta_1, D_1$</td>
<td>$\text{Shock } a_s$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$K^M_2, K^T_2$</td>
</tr>
</tbody>
</table>

**Problem 1:**
The definitions of feasibility and efficiency are standard. In order to define an efficient allocation, the decision problem of the planner is as follows:

\[
v(\bar{K}, \bar{\omega}) = \max_{(c_1, K_1) \geq 0, D_1, \theta_1 \in [0,1], c_2^L, K_1^M, K_1^T} u(c_1) \\
+ \frac{\beta}{2} [u(c_2^H(K_1, \theta_1, D_1)) + u(c_2^L(K_1, \theta_1, D_1))] \\
\text{s.t.} \quad c_1 \leq A\bar{K} + D_1 - K_1 \tag{3} \\
c_2^L \leq \left[a_s K_2^M + AK_2^T\right] + (2\mu - a_s)(1 - \theta_1)K_1 - (1 + r)D_1 \tag{4} \\
K_2^M + K_2^T \leq \theta_1 K_1 \tag{5} \\
D_1 \leq \bar{\omega}K_1 \tag{6}
\]

where \(\beta \in (0,1]\) is a discount factor; \(\mu = \frac{1}{2}a + \frac{1}{2}a\) denotes the expected value of \(a_s\); \(u(c_1)\) is the period 1 value function and \(u(c_2^L)\) is the period 2 value function in state \(s \in \{H, L\}\) (defined above).

**Definition 1 (Efficient Allocation)** Given parameter values for \((\bar{K}, \bar{\omega}, A, \bar{a}, a, r, \beta)\), an allocation \((c_1, c_2, K_1, \theta_1, D_1, K_2^M, K_2^T)\) is efficient if it solves Problem 1.

### 3.1 Gains from risk sharing with no default

To establish a benchmark case for the potential gains from international risk-sharing, suppose that there is no option to default. Then the relevant problem for the planner is as given in Problem 1. A risk-averse representative agent would clearly prefer mean consumption to a volatile consumption. Diversification (through \(\theta_1\)) allows the country to achieve exactly this type of insurance against the productivity shock, and therefore leads to consumption risk-sharing. The following proposition establishes that the country will make full use of international risk-sharing when there is no option to default \((\lambda_2 = (1,1))\), and thereby demonstrates how international risk-sharing leads to welfare gains.
Proposition 1. Under Assumption 1, a unique efficient allocation exists. Moreover, if \((c_1^*, K_1^*, D_1^*, \theta_1^*, c_2^*, K_2^{M^*}, K_2^{T^*})\) is efficient, then \(D_1^* = \bar{\omega}K_1^*, K_2^{T^*} = 0, \theta_1^* = (1/2)\), domestic output in state \(H\) is higher than in state \(L\) but \(c_2^H = c_2^L\). Hence, at an efficient allocation, period 2 output and consumption are independent.

Proof. The proof is given in the Appendix.

Proposition 1 illustrates that when the country is able to share risks, it will fully diversify income and thereby fully insure against output shocks. To see that the country achieves full diversification, note that the Proposition establishes that the country produces only in the modern sector in period 2, and that in period 1 it allocates one half of its capital each to the home and foreign sectors. As a result, while domestic output remains volatile (subject to the shock in the modern sector), domestic consumption is constant. Hence, it is clear that welfare is improved through international risk-sharing. Moreover, the uniqueness of the efficient allocations establishes that the introduction of international risk-sharing opportunities leads to a strict ex-ante welfare gain as compared to the case when there is no option to diversify. In addition, the borrowing constraint binds as with international risk sharing, despite risky borrowing as \(r > a - 1\), there is a risk-free return on capital \(\mu - 1\) and under Assumption 1 the country will therefore exhaust the borrowing constraint. By providing insurance, international risk-sharing therefore also potentially increases investment.

4 Efficient allocation: With Default

I now allow for the country to make a discrete decision in period 2 about whether to service or default on its debt obligations. I continue with an exogenous borrowing constraint in period 1 and assume that the interest rate \(r\) is given. I also maintain Assumption 1.

Introduction of default calls for concomitant default punishment to provide a credible threat in the event of default. In addition, lenders are compensated for
a default risk through a risk premium. I next discuss the default decisions and the punishment mechanism in detail. In the following section, I introduce a bond market with a no arbitrage condition to account for the risk premium.

**Default Decisions:**
The novel feature of the model is to introduce the possibility of default in period 2 (after the realization of shocks). The default decision is denoted $\lambda_2 \in \{0, 1\}$, where $\lambda_2 = 1$ means that the country services its debts in state $s$, while $\lambda_2 = 0$ means that it defaults. Hence, if the open economy has a debt stock of $D_1$ from period 1 and chooses $\lambda_2 = 1$ it must repay $(1 + r)D_1$ in state $s$. If the country chooses $\lambda_2 = 0$ it defaults on $(1 + r)D_1$ and faces default penalties. In much of the existing literature, exclusion from future borrowing is the primary default punishment (e.g., Aguiar and Gopinath, 2005; Bai and Zhang, 2012; Cuadra and Sapiriza, 2006, 2008; Lizarazo, 2009; Yue, 2010). However, exclusion from future borrowing is not a suitable punishment in a two period model. Moreover, Bulow and Rogoff (1989) argue that exclusions from future borrowing is in general not sufficient to prevent default and therefore not sufficient to provide access to international capital for emerging economies. Empirical evidence also suggests that exclusion either does not occur or is only very short-term (see, e.g., Beers and Bhatia, 1999). Bulow and Rogoff (1989) suggest that other direct punishment mechanisms are required: “Our analysis establishes rather general conditions under which small countries cannot establish a reputation for repayment. If these conditions are met empirically, then loans to LDCs (less developed countries) are possible only if creditors have either political rights which enable them to threaten the debtor’s interests outside its borrowing relationship, or legal rights. Legal rights might include the ability to impede a country’s trade, or to seize it financial assets abroad.” I capture both of the direct mechanisms suggested by Bulow and Rogoff (1989) by assuming that there are two punishments for default.

First of all, if the country defaults all capital invested abroad is repossessed (i.e., returns on that capital do not accrue to the defaulting country). This seizure of financial assets abroad represents a simple way for debtors to recover some of the value of loans that have been defaulted on. However, if the country does not
invest much capital abroad, repossession of foreign assets is not a strong deterrent to default. I therefore also allow for a country that defaults on debts to be punished in terms of trade sanctions. Specifically, I assume that in period 2 the country retains the option to produce in the traditional sector only. If the country defaults, trade sanctions are imposed and it is not able to participate in international trade, and therefore cannot produce in the modern sector. Hence, a defaulting country is forced to produce using the traditional sector. This impediment to trade represents a default punishment because, by assumption, productivity in the modern sector dominates productivity in the traditional sector in both states of the world. It also captures, in a stylized way, the empirical observation that productivity – especially in exporting sectors – usually falls dramatically after defaults events.\textsuperscript{12}

Hence, the borrower country faces the following trade-off: If it defaults it saves on the payment of debts and therefore retains a higher quantity of capital to use in domestic production. However, default also leads to repossession of foreign assets, and to trade restrictions which reduce the marginal return per unit of capital employed domestically.

Intuitively, default can also be viewed as an alternative insurance mechanism for the agent that faces a bad productivity shock. In the event of an incomplete market (for example, with 2 states and 1 asset for insurance), default provides an additional asset to the agent. As discussed in Dubey et al. (2005), agents may use default as an alternative way to insure themselves when realize of a bad productivity shock as default may be more fitting to the needs of the agent. Similarly, in the current setting, an agent faces a trade-off between diversification mechanism through ex-ante capital allocation in a perfectly negatively correlated production process in the foreign sector and defaulting ex-post on external debt. However, the value of default on external debt is linked to the ex-ante capital allocation decisions as well as ex-post realization of the state. It is most fitting and valuable for the agent to default on her/his external debt if the agent does not avail the ex-ante diversification and

\textsuperscript{12}Martinez and Sandleris (2011) presents empirical evidence that sovereign defaults are associated with a decline in trade and productivity.
realizes a bad state.

As it will be clear from the main result, the equilibrium with default in our setting is also closer to Dubey et al. (2005)'s example 2 which illustrates that despite having receipts on hand to fulfill the promise of repayment to lenders, the borrower defaults if realizes a bad state of the economy and therefore such a default can be classified as a strategic default.

**Risk Premium:** Intuitively, the default option becomes valuable as borrowing increases. But it is clear that if international lenders had rational expectations regarding the default decisions of the borrower country, they would accept default risk only if they are rewarded with an appropriate risk-premium.

In this section, I also augment the benchmark model to allow for a risk-premium to be paid to international investors in the presence of a default option. I study an augmented model in which the country borrows by selling bonds to international investors and impose a no arbitrage condition on the price of bonds to account for the possibility of default. I then look for an allocation, a bond price and a borrowing constraint such that (1) the open economy makes optimal choices given the borrowing constraint and bond price, and (2) given the default decisions of the borrower country, international investors are indifferent between the purchase of bonds or the use of international capital markets at the given rate of interest \( r \) (the no-arbitrage condition).

**Timeline:** A timeline summarizing the order in which decisions are made is given in Figure 3.

**Figure 3: Timeline Risk Adjusted Equilibrium**
Problem 2:

\[ v(\bar{K}, \bar{\omega}) = \max_{(c_1, K_1) \geq 0, D_1, \theta_1 \in [0,1], c_2^s, K_2^M, K_2^{Ts}} u(c_1) \]

\[ + \frac{\beta}{2} \left[ u(c_2^H(K_1, \theta_1, D_1)) + u(c_2^L(K_1, \theta_1, D_1)) \right] \]

s.t. \[ c_1 \leq A\bar{K} + PD_1 - K_1 \] \hfill (7)

\[ c_2^s \leq [\lambda_2^s a_s K_2^M + AK_2^{Ts}] + \lambda_2^s (2\mu - a_s)(1 - \theta_1)K_1 - \lambda_2^s (1 + r)D_1 \] \hfill (8)

\[ K_2^M + K_2^{Ts} \leq \theta_1 K_1 \] \hfill (9)

\[ D_1 \leq \bar{\omega}K_1 \] \hfill (10)

Comparing problem 2 with problem 1, shows two main differences. Default decision denoted by \( \lambda_2^s \) is introduced and risk premium captured via \( P \geq 0 \) is added.

**No-Arbitrage-Condition:**

I assume that the number of international investors is large relative to the size of the open economy, and that international investors are well diversified so that they behave risk-neutral in their lending decisions. Lenders can either buy bonds from the open economy at a price \( P \), or invest money in the international capital markets and obtain a risk-free rate of interest \( r \). International lenders can condition their choices on the price of bonds issued by the small open economy, the initial capital stock \( \bar{K} \), as well as the borrowing constraint \( \bar{\omega} \). Note that the borrowing constraint and the initial capital stock are both observable at the time the country issues bonds, so the basic assumption here is one of common knowledge regarding basic economic primitives. The bond prices should then satisfy the following no-arbitrage condition.

\[ P(1 + r) = \frac{1}{2}(1 + r)\lambda_2^H + \frac{1}{2}(1 + r)\lambda_2^L \] \hfill (11)

The no-arbitrage condition therefore determines an equilibrium price of bonds in the sense that (1) the borrower country acts optimally given the price, and (2) international investors are indifferent between the purchase of bonds (at the given
Table 1: No Arbitrage Condition

<table>
<thead>
<tr>
<th>Lender’s Options</th>
<th>Invest</th>
<th>Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lend on the international market</td>
<td>$P$</td>
<td>$P(1 + r)$</td>
</tr>
<tr>
<td>Lend to country that never defaults</td>
<td>$P$</td>
<td>$(1 + r)$</td>
</tr>
<tr>
<td>Lend to country that defaults in one state only</td>
<td>$P$</td>
<td>$\frac{1}{2}(1 + r) + \frac{1}{2}0 = \frac{1}{2}(1 + r)$</td>
</tr>
<tr>
<td>Lend to a country that always defaults</td>
<td>$P$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

To motivate the no-arbitrage condition further, suppose that the price of a bond issued by the borrower country is $P$. An international investor then has two choices in period 1: (1) The investor can lend $P$ units of period 1 capital to the international market, for which (s)he will obtain a return of $P(1 + r)$ in period 2. Alternatively, (2) the lender can buy 1 bond from the open economy and obtain a return of $(1 + r)$ in each state in which the country services debts, and 0 in each state in which the borrower country defaults. Hence, the lender’s returns depend on the borrower country’s default decisions. The possible outcomes for a risk-neutral international lender are summarized as in the table 1.

Since the initial capital stock $\bar{K} > 0$ and the borrowing constraint $\bar{\omega}$ are known, international investors can correctly predict the default behavior of the open economy. A risk-adjusted equilibrium therefore requires that for any borrowing constraint and bond price $P$, the optimal default decision of the open economy equates the return identified in the corresponding row from Table (1) with the risk-free return in international markets $P(1 + r)$. So, for example, if the borrowing constraint and bond price leads the country to assume a level of debt to capital at which it will default in neither state, then in the equilibrium it must be that the price of the bond solves

$$P(1 + r) = (1 + r)$$  \hspace{1cm} (12)

$$\Rightarrow \quad P = 1.$$  \hspace{1cm} (13)
If the borrowing constraint and price of the bond lead the country to default in exactly one state of the world, then in the equilibrium it must be that the bond price solves

\[
P(1 + r) = \frac{1}{2}(1 + r)
\]

\[\Rightarrow P = \frac{1}{2}
\]

Finally, it is clear that there can be no equilibrium in which the country defaults on bonds that have a strictly positive value in both states of the world (since if the country defaults in both states of the world, the no-arbitrage condition implies that price of bonds will be zero). Hence, with a large number of investors, the no arbitrage condition is required to ensure that supply for bonds equals the demand for bonds. If investors preferred bonds to investments in the capital market, demand would be infinite and therefore exceed supply. If investors preferred investments in the capital market to the purchase of bonds, demand would be zero and any positive bond issue would lead to excess supply. Although, I do not model the dynamics of price adjustment implicitly, the no arbitrage condition should therefore simply be interpreted as a market clearing condition that determines the bond price given rational expectations about default behavior.

4.1 Risk Adjusted Equilibrium

**Definition 2 (Risk-adjusted equilibrium)** For a given set of parameters \((\bar{a}, a, A, r, \beta, \bar{K}, \omega)\), an allocation \((K_1, D_1, \theta_1, \lambda_2, K^M_2, K^T_2)\), and a bond price \(P\) constitute a Risk-adjusted equilibrium (RAE) relative to \(\bar{K} > 0\) if \((K_1, D_1, \theta_1, \lambda_2, K^M_2, K^T_2)\) solves Problem 2 and the no arbitrage condition is satisfied:

\[
P(1 + r) = \frac{1}{2}(1 + r)\lambda^H_2 + \frac{1}{2}(1 + r)\lambda^L_2
\]

**Proposition 2** Suppose that Assumption 1 is satisfied. Let \(\omega_1\) be the cut-off value depending on \((\bar{a}, a, A, r, \gamma)\) and \(\omega_1 \in (0, \frac{a}{1+r})\) and the following hold:
1. For any $\bar{K} > 0$ and any $\bar{\omega} \in [0, \omega_1]$ there exists a unique risk-adjusted equilibrium with an allocation $K_1^* > 0$, $D_1 = \bar{\omega}K_1^*$, $\theta_1^* = 1/2$, $\lambda_2^* = (1, 1)$, $K_2^{M*} = ((1/2)K_1^*, (1/2)K_1^*)$, $K_2^{T*} = (0, 0)$ and a price $P = 1$. Call these Type I equilibria.

2. For any $\bar{K} > 0$ and any $\bar{\omega} \in (\omega_1, \frac{\omega_1}{1+r}]$ there exists a unique risk adjusted equilibrium with an allocation $K_1^* > 0$, $D_1 = \bar{\omega}K_1^*$, $\theta_1^* = 1$, $\lambda_2^* = (1, 0)$, $K_2^{M*} = (K_1^*, 0)$, $K_2^{T*} = (0, K_1^*)$ and a price $P = 1/2$. Call these Type II equilibria.

**Proof.** The proof is given in the Appendix.

The Type I equilibrium identified in Proposition 2 involves full risk-sharing, limited borrowing and no default. However, Proposition 2 also demonstrates that when the borrowing constraint is weaker, there exists a different RAE in which the country borrows more (relative to its period 2 capital stock), services debts in the high state of the world and defaults in the low state of the world. Moreover, the open economy does not participate in risk-sharing. Hence, there is a trade-off between risk-sharing through diversification of capital ownership (which dominates in a Type I equilibrium) and the default option (which dominates in a Type II equilibrium). The optimal decision is determined by the exogenous borrowing constraint.

The welfare comparison between Type I and Type II equilibrium is subtle. It is possible that the representative agent in the small open economy would strictly prefer the equilibrium of Type I (with risk-sharing) over the equilibrium of Type II (with default). However, from an international capital allocation perspective, the equilibrium of Type II (with risk-taking and default) is more efficient. This follows because, under Assumption 1, capital invested in the modern sector of the open economy is (in expectation) more productive than the alternative uses to which international capital can be put. The equilibrium with default involves a greater allocation of capital to the open economy, and it is therefore inefficient to restrict the allocation of capital to the borrowing country. Restrictions on lending are necessary only because the country may decide to default on debt obligations otherwise. The
existence of different types of RAE, therefore identifies the potential for a trade-off between the welfare of the open economy and the efficient allocation of international capital. Under the right assumptions on parameter values, the representative agent in the open economy would prefer the equilibrium with greater risk-sharing, while from an international capital allocation perspective the equilibrium with risk-taking is desirable because it allocates more capital to the country where it will be used most productively. This trade-off comes primarily from the assumption that the representative agent is risk-averse, while international investors are risk-neutral.

On the other hand, there also exist parameter values under which the open economy strictly prefers the equilibrium in which it pays a premium to borrow capital from international investors, does not share risks, exercises its default option in the bad state of the world and therefore faces volatile consumption (from an ex-ante perspective). Since the risk-sharing regime in a Type I equilibrium involves a stronger borrowing constraint, Type II equilibrium implies higher welfare when higher borrowing and a greater value of default dominates the advantages of paying a lower premium for credit and sharing risk through international diversification.

5 Conclusion

This paper has studied a simple 2 period model of an open economy that has ex-ante opportunities to allocate capital in a negatively correlated foreign sector to share risk, and an ex-post policy decision about default on external debts. The model highlights a general trade-off between risk-sharing and the option value of default. This trade-off can lead to endogenous risk-taking: Even if the country is risk-averse, full insurance against productivity shocks is possible and risk premium is to be paid for default risk to the lenders, the optimal plan may keep consumption volatile because the exposure to shocks is crucial to the option value of default.

The model relates the trade-off between ex-ante capital allocation for diversification and ex-post default options to the external debt to capital ratio, and identifies a source of threshold effects that enters into consumption risk-sharing incen-
tives through the credit-risk on external debt. The model therefore has quantifiable implications that can be used to guide and organize empirical analysis on two well-documented puzzles regarding emerging economies: (1) the apparent lack of international consumption risk-sharing in the wake of financial integration, and (2) the history of “serial default” on sovereign and external debt obligations. The analysis of this paper therefore provides one way to understand theoretically why there may be a direct link between the diverse risk-sharing and default behaviors of different emerging economies.
References


A Appendix

**Proof of Proposition 1.** The unique efficient allocation is found by backward induction. Fix any \((K_1, D_1, \theta_1)\). By the assumption \(a_s \in \{\bar{a}, a\} > A\), period 2 utility from \(K_T^2 = (0, 0)\) and \(K^M_2 = (K_1, K_1)\) is always at least as great as the utility from any other period 2 capital allocation. Hence, consider the following simplified value function in state \(s \in \{H, L\}\):

\[
\begin{align*}
\max \{K_1 \geq 0, D_1, \theta_1 \in [0, 1]\} & \quad u(AK + D_1 - K_1) \\
& + \frac{\beta}{2} \left[ u(\bar{a}\theta_1 K_1 + (2\mu - \bar{a})(1 - \theta_1)K_1 - (1 + r)D_1) \\
& + u(a\theta_1 K_1 + (2\mu - a)(1 - \theta_1)K_1 - (1 + r)D_1) \right] \\
& \text{s.t} \quad D_1 \leq \bar{a}K_1 \\
& \text{(17)} \\
& \text{(18)}
\end{align*}
\]
We can solve for an optimal \( \theta_1 \) independently of the \((K_1, D_1)\) decision because of the concavity of \( u \). Fix any \((K_1, D_1)\) (feasible) and define

\[
A : = \theta_1 [(\bar{a} - a)K_1 + \bar{a}K_1 - (1 + r)D_1] + (1 - \theta_1) [\bar{a}K_1 - (1 + r)D_1] \quad (19)
\]

\[
B : = \theta_1 [(\bar{a} - a)K_1 + \bar{a}K_1 - (1 + r)D_1] + (1 - \theta_1) [\bar{a}K_1 - (1 + r)D_1] \quad (20)
\]

For a given \((K_1, D_1)\) and an arbitrary \( \theta_1 \in [0, 1] \) note that period 2 expected utility is given by \((1/2)u(A) + (1/2)u(B)\), while expected utility for \( \theta = 1/2 \) is given by \( u[(A + B)/2] \). Since \( u \) is concave,

\[
u \left( \frac{A + B}{2} \right) \geq \frac{1}{2} u(A) + \frac{1}{2} u(B) \quad (21)
\]

and the inequality is strict whenever \( K_1 > 0 \). It therefore remains to show that with \( K_2^T = (0, 0) \) and \( \theta_1 = 1/2 \) there exist unique optimal solutions for \( K_1 \) and \( D_1 \) in which \( K_1 > 0 \) and \( D_1 = \bar{\omega} K_1 \). With \( \theta_1 = \frac{1}{2} \) the problem can be written as:

\[
\max_{\{K_1 \geq 0, D_1\}} u(A\bar{K} + D_1 - K_1) + \beta[u(\mu K_1 - (1 + r)D_1)] \quad (22)
\]

\[
s.t \quad D_1 \leq \bar{\omega} K_1
\]

We now argue that \( K_1 = 0 \) is always strictly dominated. Suppose \( K_1 = 0 \) then by the constraint \( c_2 \geq 0 \) this implies \( D_1 \leq 0 \). Suppose first that there is any feasible plan with \( D_1 = 0 \), then \( c_2 = 0 \) and for \( \epsilon > 0 \) and sufficiently small \( K_1 = \epsilon \) is feasible. Moreover, by the Inada condition on \( u \), for \( \epsilon \) sufficiently small, \( K_1 = \epsilon \) strictly dominates \( K_1 = 0 \). Hence, \( K_1 = 0 \) and \( D_1 = 0 \) is strictly dominated by a plan with \( K_1 > 0 \) and \( D_1 \leq 0 \). Now suppose \( K_1 = 0 \) and \( D_1 < 0 \) but feasible, with corresponding \( c_1 \geq 0 \). Then consider the alternative plan \((\bar{K}_1, \bar{D}_1, \bar{c}_1) = (-D_1, 0, c_1)\). This plan is feasible since \((K_1, D_1, c_1)\) is feasible since \( \mu > (1 + r) \), this plan strictly dominates \((K_1, D_1)\) because it leads to strictly greater \( c_2 \) in both states. Hence,
$K_1 = 0$ is always strictly dominated by a plan with $K_1 > 0$.

Now suppose that $K_1 > 0$, then we argue that $D_1 < \bar{\omega}K_1$ is strictly dominated by $\hat{D}_1 = \bar{\omega}K_1$. To see this, suppose that there is a feasible plan with $D_1 = \bar{\omega}K_1 - \epsilon$ for some $\epsilon > 0$ and period 1 consumption of $c_1 \geq 0$. Then the plan $(\hat{K}_1, \hat{D}_1, \hat{c}_1) = (K_1 + \epsilon, D_1 + \epsilon, c_1)$ is feasible. Since $c_1$ remains unchanged period 1 utility remains unchanged. However, $c_2$ increases by $(\mu - (1 + r))\epsilon > 0$ (by Assumption 1). Hence, for $K_1 > 0$, $D_1 < \bar{\omega}K_1$ is strictly dominated by $\hat{D}_1 = \bar{\omega}K_1$.

It therefore remains to show that there exists a unique $K_1 > 0$ that solves the following optimization problem:

$$\max_{K_1 > 0} u(A\bar{K} - (1 - \bar{\omega})K_1) + \beta u((\mu - \bar{\omega}(1 + r))K_1)$$

It is straightforward to verify that the objective function is continuously differentiable and strictly concave and so the following first order condition is necessary and sufficient for a solution to (23):

$$f(K_1) : = \beta (\mu - \bar{\omega}(1 + r))u'((\mu - \bar{\omega}(1 + r))K_1) - (1 - \bar{\omega})u'(A\bar{K} - (1 - \bar{\omega})K_1) = 0$$

Note that

$$f'(K_1) = \beta u''(\mu K_1 - (1 + r)\bar{\omega}K_1)(\mu - (1 + r)\bar{\omega})^2 > 0$$

$$+ (1 - \bar{\omega})^2 u''(A\bar{K} - (1 - \bar{\omega})K_1) < 0$$

$$\lim_{K_1 \to 0} f(K_1) > 0$$

$$\lim_{K_1 \to \frac{AK}{1 - \bar{\omega}}} f(K_1) < 0$$
Hence, by the intermediate value theorem, there exists a unique \( K_1^* > 0 \) that solves (25). It follows that \( K_1^* > 0, D_1 = \bar{\omega}K_1^*, \theta_1 = (1/2), K_2^T = (0, 0) \) and \( K_2^M = (K_1^*, K_1^*) \) is the unique efficient allocation. ■

**Proof of Proposition 2.** Suppose \( P = 1 \), and consider Problem 2:

\[
v(\bar{K}, \bar{\omega}) = \max_{\{(c_1, K_1) \geq 0, D_1, \theta_1 \in [0, 1], c_2^s, K_2^M, K_2^T \}} u(c_1) \\
+ \frac{\beta}{2} [u(c_2^H(K_1, \theta_1, D_1)) + u(c_2^L(K_1, \theta_1, D_1))]
\]

s.t. \[
c_1 \leq A\bar{K} + D_1 - K_1 \\
c_2^s \leq [\lambda_2^s a_s K_2^M + AK_2^T] + \lambda_2^s (2\mu - a_s)(1 - \theta_1)K_1 - \lambda_2^s (1 + r)D_1 \\
K_2^M + K_2^T \leq \theta_1 K_1 \\
D_1 \leq \bar{\omega}K_1
\]

We can solve this problem by backward induction.

In period 2 there are four possible default choices. By the assumption \( \bar{a} > a > A \) each default choice immediately determines an optimal period 2 capital allocation. Hence, there are four possible optimal plans in period 2:

1. \( \lambda_2 = (1, 1), K_2^M = (K_1, K_1), K_2^T = (0, 0) \);
2. \( \lambda_2 = (0, 0), K_2^M = (0, 0), K_2^T = (K_1, K_1) \);
3. \( \lambda_2 = (1, 0), K_2^M = (K_1, 0), K_2^T = (0, K_1) \);
4. \( \lambda_2 = (0, 1), K_2^M = (0, K_1), K_2^T = (K_1, 0) \).

We next study the optimal period 1 decision corresponding to each of the four cases.

Begin with case (1) and observe that by Proposition 1 we have that for any \((\bar{K}, \bar{\omega})\) there exists a \( K_1^*(1) > 0 \) such that \( K_1^*(1) > 0, D_1 = \bar{\omega}K_1^*(1), \theta_1 = (1/2) \) are the optimal period 1 choices.
Now consider case (2). In this case $\theta_1 = 1$ clearly dominates any $\theta_1 < 1$, and dominates strictly if $K_1 > 0$. We can use the same argument used in the proof of Proposition 1 to show that $K_1 > 0$ strictly dominates $K_1 = 0$. Moreover, using $\theta_1 = 1$, the constraint $D_1 = \bar{\omega}K_1$ clearly binds. It then follows as for case 1 that there exists $K_1^*(2) > 0$ such that $K_1^*(2) > 0$, $D_1 = \bar{\omega}K_1^*(2)$, $\theta_1 = 1$ are the optimal period 1 choices.

Now consider case (3). Again, $\theta_1 = 1$ clearly dominates any $\theta_1 < 1$, and dominates strictly if $K_1 > 0$. It again follows from the same argument as above that $K_1 > 0$ strictly dominates $K_1 = 0$. Moreover, the constraint $D_1 = \bar{\omega}K_1$ must bind. Since the constraint binds in case (1) where in state $H$ and $L$, the return on capital is $\mu$, and in case (3) the return on capital is $\bar{a} > \mu$, and in state $L$ there is a default. It then follows as for case 1 that there exists $K_1^*(3) > 0$ such that $K_1^*(3) > 0$, $D_1 = \bar{\omega}K_1^*(3)$, $\theta_1 = 1$ are the optimal period 1 choices.

Finally, consider case (4). Fix any $\theta_1 \in [0,1]$ and observe that by the same arguments as previously, $K_1 > 0$ and the constraint $D_1 = \bar{\omega}K_1$ binds. Now observe that in period 1, for any $K_1 > 0$ and $D_1 = \bar{\omega}K_1$ the expected period 2 utility under case (4) is

$$\frac{1}{2} u([A\theta_1]K_1) + \frac{1}{2} u\left( ([a\theta_1 + \bar{a}(1 - \theta_1)] - (1 + r)\bar{\omega}) K_1 \right) \quad (32)$$

For $\theta_1 = 1$, the expected period 2 utility under case (3) is

$$\frac{1}{2} u(AK_1) + \frac{1}{2} u( [\bar{a} - (1 + r)\bar{\omega}]K_1) \quad (33)$$

Clearly, for any $\theta_1 \in [0,1]$ the expected utility in (32) dominated by the expected utility in (33), and dominated strictly whenever $K_1 > 0$. Hence, case (4) will not affect period 1 decisions.

We are therefore left with three cases, each of which directly determines an optimal $\theta_1$ decision, and in each of which $K_1 > 0$ and the collateral constraint binds.

We next determine which case dominates from a period 1 perspective. For this, we use the fact that utility is multiplicatively separable, so that each case corresponds
to a unique optimal choice of $\theta_1$ independently of $K_1$. This allows us to compare the respective expected period 2 utilities only in terms of parameters. Corresponding to each of the three remaining cases and for any $K_1 > 0$, these are therefore:

Case 1: $u((\mu - (1 + r)\bar{\omega})u(K_1))$  \hspace{1cm} (34)

Case 2: $u(A)u(K_1)$  \hspace{1cm} (35)

Case 3: $\left[\frac{1}{2}u((\bar{a} - (1 + r)\bar{\omega})) + \frac{1}{2}u(A)\right] u(K_1)$  \hspace{1cm} (36)

We now compare these three expected utilities to determine the $\omega_1$ and $\omega_2$.

- **Case 1 < Case 2:** By the assumption that $u$ is strictly increasing, this holds if and only if

  \[
  A > \mu - (1 + r)\bar{\omega} \hspace{1cm} (37)
  \]

  \[
  \bar{\omega} > \frac{\mu - A}{1 + r} := \omega_3 \hspace{1cm} (38)
  \]

- **Case 3 < Case 2:** By the assumption that $u$ is strictly increasing, this holds if and only if

  \[
  A > (\bar{a} - (1 + r)\bar{\omega}) \hspace{1cm} (39)
  \]

  \[
  \bar{\omega} > \frac{\bar{a} - A}{1 + r} := \omega_2 \hspace{1cm} (40)
  \]

- **Case 1 > Case 3:** It is not possible to get a closed form solution, but we show that a unique $\omega_1$ exist between 0 and $\omega_3$ such that Case 1 > Case 3 if and only if $\bar{\omega} < \omega_1$. Define $f(\omega)$ as:

  \[
  f(\omega) = u(\mu - (1 + r)\omega) - \frac{1}{2}u(\bar{a} - (1 + r)\omega) - \frac{1}{2}u(A) \hspace{1cm} (41)
  \]
First observe that
\[
\frac{\partial f(\omega)}{\partial \omega} = -(1 + r)u'(\mu - (1 + r)\bar{\omega}) - \frac{1}{2}(-1 - \mu)u'(\bar{a} - (1 + r)\bar{\omega})
\]
\[
= -(1 + r)\left[u'(\mu - (1 + r)\bar{\omega}) - \frac{1}{2}u'(\bar{a} - (1 + r)\bar{\omega})\right] < 0
\]
\[
\leq 0
\]
(42)

Now note that
\[
f(0) = u(\mu) - \frac{1}{2}u(\bar{a}) - \frac{1}{2}u(A)
\]
(45)

By strict concavity of \(u\),
\[
u(\mu) > \frac{1}{2}u(\bar{a}) + \frac{1}{2}u(a)
\]
(46)
\[
> \frac{1}{2}u(\bar{a}) + \frac{1}{2}u(A)
\]
(47)

Hence, \(f(0) > 0\). Next note that
\[
f\left(\frac{\mu - A}{1 + r}\right) = u(A) - \frac{1}{2}u(\bar{a} - (\mu - A)) - \frac{1}{2}u(A)
\]
\[
= \frac{1}{2}\left[u(A) - u(A + (\bar{a} - \mu))\right]
\]
(48)
(49)

Since \(\bar{a} > \mu\) and \(u\) is strictly increasing, it follows that \(f((\mu - A)/(1 + r)) < 0\). Hence, by the continuity of \(u\), there exists a unique \(\omega_1 \in (0, \omega_2)\) such that \(f(\omega) > 0\) if and only if \(\omega < \omega_1\) and \(f(\omega) = 0\) if and only if \(\omega = \omega_1\).

- Now observe that we have \(\omega_1, \omega_2, \omega_3\) such that \(0 < \omega_1 < \omega_3 < \omega_2\). Also, for \(\bar{\omega} \in (0, \omega_1)\) Case 1 strictly dominates Case 2 and Case 3; for \(\bar{\omega} = \omega_1\) Case 1 and Case 3 lead to the same expected utility and both dominate Case 2.

When \(\bar{\omega} \leq \omega_1\), we have a risk adjusted equilibrium (RAE) with \(\lambda(1, 1), P = 1\). If \(\bar{\omega} > \omega_1\), then no RAE with \(P = 1\) exists. To show that first recall that \(0 < \omega_1 < (\mu - A)/(1 + r)\). By Assumption 1, \(\bar{a} < \bar{a} + 2A\), which implies that \(\mu < \bar{a} + A\) and
therefore $\omega_1 < a/(1 + r)$.

Now start with part (1) of the Proposition. Fix $\bar{K} > 0$ and let $\bar{\omega} \in [0, \omega_1]$. Suppose that the price $P = 1$. Moreover, Assumption 1 implies that $\mu > (1 + r)$ (from $\bar{a} > 2(1 + r)$) and it therefore follows from Proposition 1 that there exist a unique $K_1^* > 0$, such that $(K_1^*, D_1^* = \bar{\omega}K_1^*, \theta_1^* = 1/2, \lambda_2^* = (1, 1), K_M^* = ((1/2)K_1^*, (1/2)K_1^*), K_T^* = (0, 0))$ is the unique optimal plan for the borrower country. Moreover, given $\lambda_2^* = (1, 1)$ the no arbitrage condition implies that $P = 1$. Hence, this is a Type I risk adjusted equilibrium.

Now note that for any $P \geq 0$ the collateral constraint implies that it is never optimal for the borrower country to default since the period 2 constraint is identical to the period 2 in Problem 3, and with a debt to capital ratio $D_1/K_1 < \omega_1$ it follows from the proof of Proposition 1 that default in period 2 is never optimal. As a result, an optimal plan for the borrower country always involves $\lambda_2 = (1, 1)$. Hence, any price $P \neq 1$ can not occur in a risk adjusted equilibrium. Given the tie breaker rule for $\bar{\omega} = \omega_1$, this proves uniqueness.

Now look at part (2) of the Proposition. First suppose that $P = 1/2$. To show that the collateral constraint will bind at an optimal plan, first consider the case when the default decision in period 2 is $\lambda_2 = (1, 0)$. Suppose for sake of contradiction that $(c_1, K_1, D_1) \geq 0$ is part of a feasible plan in which $D_1 = \bar{\omega}K_1 - \epsilon$ for some $\epsilon > 0$ (hence, the collateral constraint does not bind). Consider an alternative plan in which period 1 choices have changed to $(\hat{c}_1, \hat{K}_1, \hat{D}_1) = (c_1, K_1 + (\epsilon/2), D_1 + \epsilon)$. Note that at a price of $P = 1/2$ this plan is feasible because the plan $(c_1, K_1, D_1)$ was feasible by assumption. Under the new plan consumption in period 1 is unchanged and the change in period 2 consumption in state $H$ is $\bar{a}(\epsilon/2) - (1 + r)\epsilon$. By Assumption 1, $\bar{a} > 2(1 + r)$ and therefore the last term is strictly positive. The change in period 2 consumption in state $L$ is $A(\epsilon/2) > 0$. Hence, expected utility under the plan $(\hat{c}_1, \hat{K}_1, \hat{D}_1)$ is strictly greater because $u$ is strictly increasing.

By an analogous argument, the collateral constraint binds when the default decision in period 2 is $\lambda_2 = (1, 1)$. When the default decision is $\lambda = (0, 0)$ the collateral constraint necessarily binds, since additional borrowing can always be used to in-
crease period 1 utility without any decrease in period 2 utility (due to default).

The collateral constraint therefore binds for all relevant (i.e., not always dominated) period 2 default decisions, and the collateral constraint must therefore bind at an optimal plan.

However, since the period 2 constraint is identical in Problem 1 and Problem 2, it follows just as in Proposition ?? that when the collateral constraint binds and \( \bar{\omega} \in (\omega_1, \omega_2) \), there exists a unique \( K_1^* > 0 \) such that the unique optimal plan of the borrower country is \((K_1^*, D_1^* = \bar{\omega}K_1^*, \theta_1^* = 1, \lambda_2^* = (1, 0), K_M^* = (K_1^*, 0), K_T^* = (0, K_1^*))\). Also, given \( \lambda_2^* = (1, 0) \) the no arbitrage condition implies that \( P = 1/2 \). Hence, for \( \bar{\omega} \in (\omega_1, \omega_2) \), this is a Type II risk adjusted equilibrium.

To show uniqueness of the RAE found above for \( \bar{\omega} \in (\omega_1, \omega_2) \), first suppose that \( P > 1/2 \). Then it is clear that at any optimal plan the collateral constraint binds and therefore \( \lambda_2 = (1, 0) \). It follows that \( P \) is not consistent with the no arbitrage condition. Now suppose that \( P < 1/2 \). This can be consistent with the no arbitrage condition only if \( P = 0 \) and \( \lambda_2 = (0, 0) \). But for \( \bar{\omega} \leq \omega_2 \) it follows from Proposition ?? that the representative agent is always at least as well-off not defaulting in the high state. Hence, \( P = 1/2 \) is the unique price at which there exists a risk adjusted equilibrium when \( \bar{\omega} \in (\omega_1, \omega_2) \).

Finally, note that under Assumption 1, it is not the case that \( \omega_2 \leq a/(1+r) \). Since from Assumption 1, \( \bar{a} > 2a > a + A \), it follows that \( \omega_2 := (\bar{a} - A)/(1+r) > a/(1+r) \). As a result, for the range of \( \bar{\omega} \) considered, \( \bar{\omega} \in a/(1+r) \), the two types of RAE equilibria found above are both mutually exclusive and collectively exhaustive. ■